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Interpolating Polynomial Macro-Elements with Tension Properties

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Abstract. In this paper we present the construction of nine-parameter polynomial macro elements, based on the classical Powell-Sabin split, which can be connected to form a C^1 surface. Variable degrees, which act as independent tension parameters, are associated with any vertex of the triangulation, i.e. to any interpolation point.

§1. Introduction

Among the several approaches for avoiding extraneous inflection points in interpolating functions, the so called tension methods are the oldest and probably the most famous ones. Basically they consist of C^k , $k \geq 1$, piecewise functions depending on a set of parameters which are selected in a local or global way to control the shape of the interpolants, stretching their patches between data points.

Piecewise polynomial splines with variable degrees have turned out to be a useful alternative to classical (exponential or rational) tension methods, and have been successfully applied both in free-form design and in constrained interpolation of spatial data [1]. On the other hand, a limited number of methods for constrained interpolation of bivariate scattered data are at present available, and very few of them offer the possibility of controlling the shape via tension parameters (see for example [2,3] and references quoted therein).

Let a set of scattered points (x_i, y_i, f_i) , $i = 0, 1, \dots, N$, be given, and suppose they have been associated with a proper triangulation \mathcal{T} . We are interested in local methods where the polynomial pieces of a spline are determined one triangle at time using only local data. Such methods are called macro-element methods. The aim of the paper is to describe a new class of variable degree polynomial triangular macro-elements, and to show their tension properties. These polynomial patches, which are based on the classical Powell-Sabin split of a triangle, can be connected to form a C^1 interpolating function.

Typically, the main drawback of macro-element methods is in the strong influence of the triangulation on the shape of the interpolating surface, and, as far as we know, it is still not clear how to construct a “good” triangulation. Given that any interpolating scheme based on triangular macro-elements cannot be completely independent of the triangulation, we can nevertheless try to reduce this dependence. From this point of view, the main advantage over existing methods is in the fact that we have a variable degree associated to each vertex of the triangulation; therefore the modification in the shape of the interpolant is similar for all the patches around the same common vertex (see Fig. 8 right). In other words, although constructed over \mathcal{T} , the tension parameters are more related to the interpolation points than to the edges of the adopted triangulation.

It is worthwhile to anticipate that the possibly high degrees we use for our construction act only as tension parameters, and do not modify the basic structure of the macro-element. In other words, we always use a nine-parameter macro-element, and the computational complexity is almost independent of the size of the degrees used.

The scheme, being local, requires that the gradients are also known at the vertices of the triangulation. If this information is unavailable, gradients can be recovered from the data points. In the presented numerical test, gradients have been computed from the data according to the classical least square strategy.

The paper is divided into six sections. In the next section we introduce some notations. Sections 3 and 4 are devoted to the construction of the control points defining the macro-element, whose properties are briefly discussed in Section 5. We end with Section 6 where a graphical example is presented.

§2. Notation and Preliminaries

In this section we introduce some notation. To aid in comprehension, we notice that points and vectors in \mathbb{R}^2 and in \mathbb{R}^3 have been denoted by bold-faced characters unless classical notations (as for gradients) have been used. As usual, we describe the polynomial macro-element in terms of its Bernstein-Bézier control points. Let \mathbf{P}_r , $r = 1, 2, 3$, be three non-collinear points in \mathbb{R}^2 , and let T denote the triangle they form. An n -degree Bernstein polynomial has the form

$$b(x, y; n) = b(u, v, w; n) := \sum_{i+j+k=n} \frac{n!}{i!j!k!} l_{ijk} u^i v^j w^k,$$

where, setting $\mathbf{P} = [x, y]^T \in \mathbb{R}^2$, $u = u(x, y)$, $v = v(x, y)$, $w = w(x, y)$, are the barycentric coordinates of \mathbf{P} with respect to the vertices of T , that is

$$\mathbf{P} = u\mathbf{P}_1 + v\mathbf{P}_2 + w\mathbf{P}_3; \quad u + v + w = 1,$$

and l_{ijk} , $i + j + k = n$, are the Bernstein ordinates of $b(., ., .; n)$.

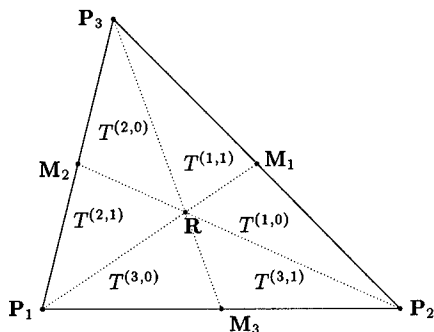


Fig. 1. Powell-Sabin split of a triangle.

Setting $x = x(u, v, w)$, $y = y(u, v, w)$, the points in \mathbb{R}^3

$$\mathbf{L}_{ijk} := \begin{bmatrix} x(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \\ y(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \\ l_{ijk} \end{bmatrix}, \quad i, j, k \geq 0, \quad i + j + k = n,$$

are called control points [4].

We finally recall the so-called Powell-Sabin split ([5]) of a given triangle, T , which consists of dividing $T := \mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ into six *mini-triangles* (Fig. 1):

$$\begin{aligned} T^{(1,0)} &:= \mathbf{P}_2\mathbf{M}_1\mathbf{R}, & T^{(2,0)} &:= \mathbf{P}_3\mathbf{M}_2\mathbf{R}, & T^{(3,0)} &:= \mathbf{P}_1\mathbf{M}_3\mathbf{R}, \\ T^{(1,1)} &:= \mathbf{M}_1\mathbf{P}_3\mathbf{R}, & T^{(2,1)} &:= \mathbf{M}_2\mathbf{P}_1\mathbf{R}, & T^{(3,1)} &:= \mathbf{M}_3\mathbf{P}_2\mathbf{R}, \end{aligned}$$

where

$$\mathbf{R} = \beta_1\mathbf{P}_1 + \beta_2\mathbf{P}_2 + \beta_3\mathbf{P}_3, \quad \beta_1 + \beta_2 + \beta_3 = 1,$$

is a point internal to T and

$$\mathbf{M}_i = (1 - \alpha_i)\mathbf{P}_{i+1} + \alpha_i\mathbf{P}_{i+2}, \quad 0 < \alpha_i < 1,$$

is a point internal to the edge of T opposite to \mathbf{P}_i . Here, and in the following, indices will be considered modulus 3.

We will denote by $\mathbf{L}_{ijk}^{(p,q)}$, $\bar{\mathbf{L}}_{ijk}^{(p,q)}$ the control points of Bernstein polynomials in the mini triangle $T^{(p,q)}$ (see below). Let the data

$$(\mathbf{P}_i, f_i = f(\mathbf{P}_i), \nabla f_i = \nabla f(\mathbf{P}_i)), \quad \mathbf{P}_i \in \mathbb{R}^2, \quad i = 1, 2, 3, \quad (1)$$

be given. As mentioned in the introduction, our goal is to construct a C^1 polynomial macro-element on T interpolating the data and having tension properties.

The macro-element will be obtained considering a Powell-Sabin split of T , and constructing in each mini triangle $T^{(p,q)}$ a Bernstein polynomial via suitable control points, $\bar{\mathbf{L}}_{ijk}^{(p,q)}$. To obtain the final control points (FCP) $\bar{\mathbf{L}}_{ijk}^{(p,q)}$, we follow two basic steps: first we construct the basic control points (BCP) $\mathbf{L}_{ijk}^{(p,q)}$, then we modify them to reach the required smoothness of the macro-element.

§3. Defining the Basic Control Points

For each vertex \mathbf{P}_i , let us consider an associated given degree

$$n_i \geq 3, \quad n_i \in \mathbb{N}.$$

Let us now describe the construction of the BCP considering for the sake of simplicity only the mini triangle $T^{(3,0)}$. See Fig. 2 for the role of indices.

First of all we assume interpolation conditions (for the position and the gradient) at \mathbf{P}_1 (see “*” in Fig. 3 left) that is

$$\begin{aligned} l_{n_1,0,0}^{(3,0)} &:= f_1, \\ l_{n_1-1,1,0}^{(3,0)} &:= f_1 + \frac{1}{n_1} \langle \nabla f_1, \mathbf{P}_1 \mathbf{M}_3 \rangle, \\ l_{n_1-1,0,1}^{(3,0)} &:= f_1 + \frac{1}{n_1} \langle \nabla f_1, \mathbf{P}_1 \mathbf{R} \rangle. \end{aligned} \quad (2)$$

In order to define the BCP around \mathbf{M}_3 , let us consider the univariate piecewise linear function, l_3 , defined along the edge $\mathbf{P}_1 \mathbf{P}_2$ having breakpoints at

$$\mathbf{P}_1, \quad \mathbf{P}_1 + \frac{1}{n_1} \mathbf{P}_1 \mathbf{M}_3, \quad \mathbf{P}_2 - \frac{1}{n_2} \mathbf{M}_3 \mathbf{P}_2, \quad \mathbf{P}_2,$$

interpolating f and its derivatives at the extremes of the edge. We define (see “ \otimes ” in Fig. 3 left)

$$\begin{aligned} l_{0,n_1,0}^{(3,0)} &:= l_3(\mathbf{M}_3), \\ l_{1,n_1-1,0}^{(3,0)} &:= l_3 \left(\mathbf{M}_3 - \frac{1}{n_1} \mathbf{P}_1 \mathbf{M}_3 \right), \\ l_{0,n_1-1,1}^{(3,0)} &:= l_3(\mathbf{M}_3) + \frac{1}{n_1} d_3, \end{aligned} \quad (3)$$

where

$$d_3 := \langle (1 - \alpha_3) \nabla f(\mathbf{P}_1) + \alpha_3 \nabla f(\mathbf{P}_2), \mathbf{M}_3 \mathbf{R} \rangle.$$

Moreover, we require that $\mathbf{L}_{1,n_1-2,1}^{(3,0)}$ (see “ \odot ” in Fig. 3 left) belongs to the plane through

$$\mathbf{L}_{1,n_1-1,0}^{(3,0)}, \quad \mathbf{L}_{0,n_1,0}^{(3,0)}, \quad \mathbf{L}_{0,n_1-1,1}^{(3,0)}.$$

Concerning the remaining control points, we assume that the not yet defined control points of the first two rows in the mini triangle parallel to $\mathbf{P}_1 \mathbf{P}_2$ (see “o” in Fig. 3 left) belong to the straight lines through the above defined control points, that is

$$\begin{aligned} \mathbf{L}_{i,n_1-i,0}^{(3,0)} &= \frac{i-1}{n_1-2} \mathbf{L}_{n_1-1,1,0}^{(3,0)} + \frac{(n_1-2)-(i-1)}{n_1-2} \mathbf{L}_{1,n_1-1,0}^{(3,0)}, \\ \mathbf{L}_{i,n_1-i-1,1}^{(3,0)} &= \frac{i-1}{n_1-2} \mathbf{L}_{n_1-1,0,1}^{(3,0)} + \frac{(n_1-2)-(i-1)}{n_1-2} \mathbf{L}_{1,n_1-2,1}^{(3,0)}, \\ i &= n_1 - 2, \dots, 2. \end{aligned} \quad (4)$$

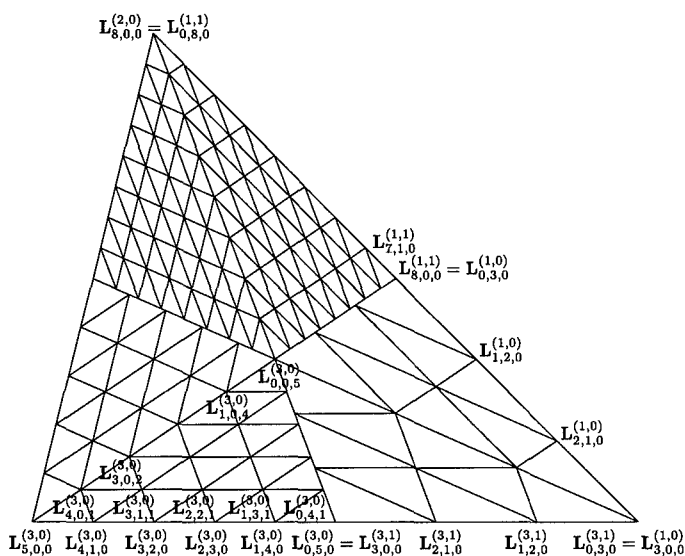


Fig. 2. The role of indices of the control points (projection onto the x, y plane) for the split of Figure 1 with $n_1 = 5, n_2 = 3, n_3 = 8$.

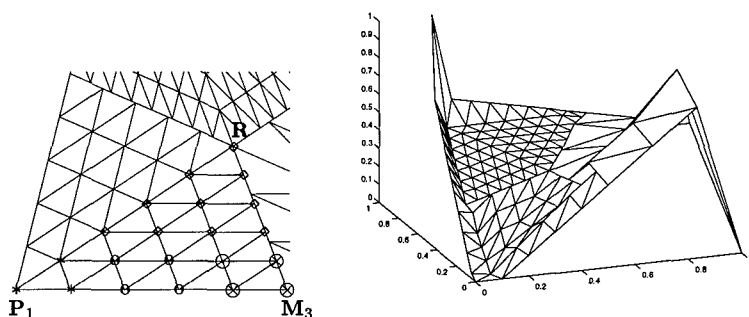


Fig. 3. The construction of the BCP for the split of Figure 1 with $n_1 = 5, n_2 = 3, n_3 = 8$. Left: projection of $L_{i,j,k}^{(3,0)}$ in the x, y plane; "*" are determined via interpolation conditions (2); "⊗" by (3); "o" by (4) and "◊" by coplanarity conditions. Right: $L_{i,j,k}^{(p,q)}$, $p = 1, 2, 3, q = 0, 1$. Note the discontinuities across the edges $M_i R$.

Moreover, we require that conditions for C^1 continuity across the edge $P_1 R$ hold [4]. Then, in particular,

$$L_{1,n_1-2,1}^{(2,1)}, L_{0,n_1-1,1}^{(2,1)}, L_{0,n_1-2,2}^{(2,1)} = L_{n_1-2,0,2}^{(3,0)}, L_{n_1-1,0,1}^{(3,0)}, L_{n_1-2,1,1}^{(3,0)},$$

lie onto the same plane.

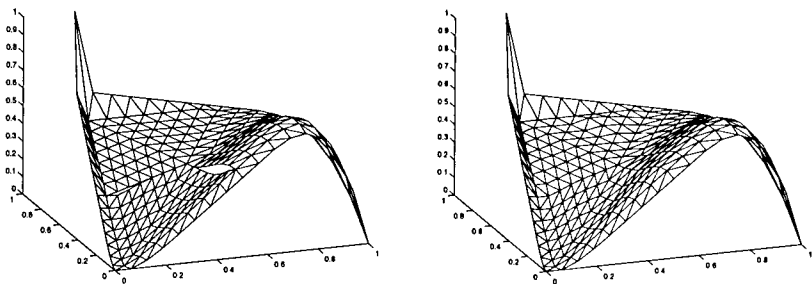


Fig. 4. *Left:* the control points $\bar{\mathbf{L}}_{i,j,k}^{(p,q)}$ after the first step, $n = 8$. *Right:* The final control points $\bar{\mathbf{L}}_{i,j,k}^{(p,q)}$ after the second step. $\mathbf{L}_{i,j,k}^{(p,q)}$ as in Fig. 3.

Finally, the central control points $\mathbf{L}_{i,j,k}^{(3,0)}$, $k \geq 2$, (see “ \diamond ” in Fig. 3 left) are assumed to lie onto the plane through

$$\mathbf{L}_{n_1-2,0,2}^{(3,0)}, \mathbf{L}_{n_2-2,0,2}^{(1,0)}, \mathbf{L}_{n_3-2,0,2}^{(2,0)}.$$

Similarly we define the BCP in the other mini triangles.

The above construction provides a polynomial macro-element which turns out to be of class C^1 across the interior edges $\mathbf{P}_i\mathbf{R}$.

Moreover, we assume that the points \mathbf{M} lying on the internal edges of the initial triangulation \mathcal{T} also lie on the straight lines joining the \mathbf{R} points of the triangles which those edges separate (see Fig. 8). This classical requirement, (2), (3), (4), and the geometry of the Powell–Sabin split ensure C^1 continuity of two macro-elements across the boundary edge $\mathbf{P}_i\mathbf{P}_{i+1}$ (see Fig. 3 right).

On the other hand, the BCP do not produce in general a continuous macro-element across the edges $\mathbf{M}_i\mathbf{R}$ unless the degrees n_j are equal (see Fig. 3 right). In order to obtain a C^1 macro-element without imposing any condition on the degrees, we modify the constructed BCP. This will be described in the next section.

§4. Obtaining a C^1 Macro Element

In this section we describe how to modify the BCP to obtain the final control points (FCP) producing a C^1 macro-element. The modified FCP will be basically obtained from the BCP via the degree-raising process in two steps.

As a first step, for each mini triangle $T^{(p,q)}$ let us compute

$$\bar{\mathbf{L}}_{i,j,k}^{(p,q)}, \quad i, j, k \geq 0, \quad i + j + k = n,$$

the control points obtained from $\mathbf{L}_{i,j,k}^{(p,q)}$, $i, j, k \geq 0$, $i + j + k = n_{p+q+1}$, by the degree-raising process ([4]) from the degree n_{p+q+1} (that is the degree associated with the mini triangle $T^{(p,q)}$) to the degree $n := \max\{n_1, n_2, n_3\}$ (see Fig. 4 left). The control points $\bar{\mathbf{L}}_{i,j,k}^{(p,q)}$ allow us to express the polynomial of degree n_{p+q+1} defined by $\mathbf{L}_{i,j,k}^{(p,q)}$ as a Bernstein polynomial of degree n .

We emphasize that, due to the geometry of the split, the control points $\bar{\mathbf{L}}_{i,j,k}^{(p,q)}$, define a macro-element which is of class C^1 across the interior edges $\mathbf{M}_p\mathbf{R}$ if and only if the control points

$$\bar{\mathbf{L}}_{1,n-1-k,k}^{(p,0)}, \bar{\mathbf{L}}_{0,n-k,k}^{(p,0)} = \bar{\mathbf{L}}_{n-k,0,k}^{(p,1)}, \bar{\mathbf{L}}_{n-k-1,1,k}^{(p,1)}, \quad k = 0, \dots, n-1,$$

are collinear. Moreover, we notice that, due to the construction of the BCP around \mathbf{M}_p and near \mathbf{R} and to the properties of the degree-raising process, the control points

$$\bar{\mathbf{L}}_{1,n-1-k,k}^{(p,0)}, \bar{\mathbf{L}}_{0,n-k,k}^{(p,0)} = \bar{\mathbf{L}}_{n-k,0,k}^{(p,1)}, \bar{\mathbf{L}}_{n-k-1,1,k}^{(p,1)}, \quad k = 0, 1, n-1,$$

lie on the same plane, then they are collinear due to the geometry of the split (see Fig. 4 left). Then in order to obtain C^1 continuity across the edge $\mathbf{M}_p\mathbf{R}$, we simply consider a second step in which we modify

$$\bar{\mathbf{L}}_{0,n-k,k}^{(p,0)}, \bar{\mathbf{L}}_{n-k,0,k}^{(p,1)}, \quad k = 2, \dots, n-2, \quad p = 1, 2, 3, \quad (5)$$

imposing that (see Fig. 4 right) they lie on the segment through

$$\bar{\mathbf{L}}_{1,n-k-1,k}^{(p,0)}, \bar{\mathbf{L}}_{n-k-1,1,k}^{(p,1)}, \quad k = 2, \dots, n-2, \quad p = 1, 2, 3.$$

§5. Properties of the Macro-Element

In this section we analyze the interpolation, smoothness and tension properties of the macro-element defined by the FCP constructed in Section 4. First of all, we notice that the construction of the macro-element is completely local: it only depends on the data at the vertices of T and on the degrees n_j associated with the vertices which are given input parameters.

Theorem 1. *The polynomial macro-element defined by the control points*

$$\bar{\mathbf{L}}_{i,j,k}^{(p,q)}, \quad p = 1, 2, 3, q = 0, 1, \quad i, j, k \geq 0, \quad i + j + k = n,$$

interpolates the data (1) and is of class $C^1(T)$. Moreover, let T be a given triangulation equipped with a classical Powell-Sabin split, and with a fixed degree associated with any vertex. Then the collection of the macro-elements corresponding to the triangles of T produces an interpolating surface of class C^1 .

Proof: The BCP defined in Section 3 produce an interpolating macro-element which is of class C^1 across the interior edges $\mathbf{P}_i\mathbf{R}$ and the boundary edges $\mathbf{P}_i\mathbf{P}_{i+1}$. After the degree-raising process, only the control points in (5) are modified in order to obtain C^1 continuity across $\mathbf{M}_i\mathbf{R}$. Since the C^1 continuity across one edge only depends on the control points lying on the

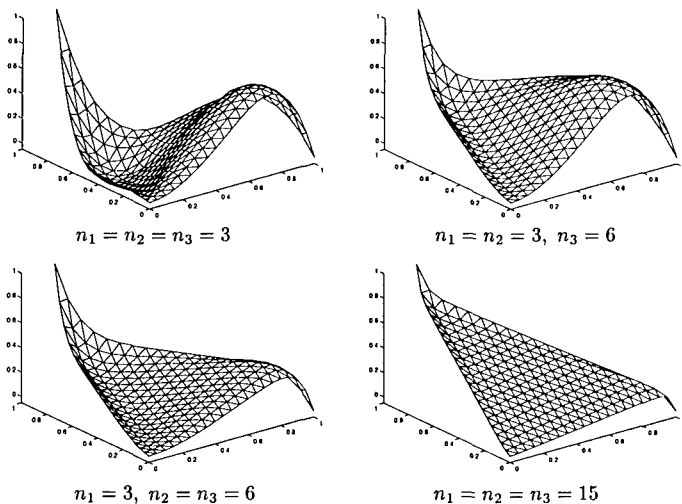


Fig. 5. The local tension effect of the degrees: vertices are numbered counter-clockwise from the origin.

first two rows parallel to that edge ([4]), the second step in Section 4 does not affect C^1 continuity across $\mathbf{P}_i\mathbf{R}$ and $\mathbf{P}_i\mathbf{P}_{i+1}$. \square

As mentioned before, the degrees n_j are free input parameters. From the construction described in Section 3, it is clear that increasing their values causes the BCP to approach the plane interpolating (\mathbf{P}_i, f_i) , $i = 1, 2, 3$. Similarly, the FCP approach the same plane because they have been obtained via a degree-raising process, that is via a convex combination of the BCP. Therefore, the same property is shared by the macro-element, due to the convex hull property of the Bernstein representation. We summarize the tension properties of our macro-element with the following theorem.

Theorem 2. *If $n_1, n_2, n_3 \rightarrow +\infty$, then the polynomial macro-element defined by the control points*

$$\bar{\mathbf{L}}_{i,j,k}^{(p,q)}, \quad p = 1, 2, 3, q = 0, 1, \quad i, j, k \geq 0, \quad i + j + k = n,$$

approaches the plane through (\mathbf{P}_i, f_i) , $i = 1, 2, 3$.

We end this section by emphasizing that the degrees act as *local* tension parameters: each degree affects the shape of the interpolating surface only around the associated vertex (see Fig. 8 right), and, as previously said, this is the main feature of this method. The increase of a degree pushes the surface to the piecewise linear interpolant around the corresponding point, giving it a cuspidal appearance. This local tension effect is clearly shown in Fig. 5.

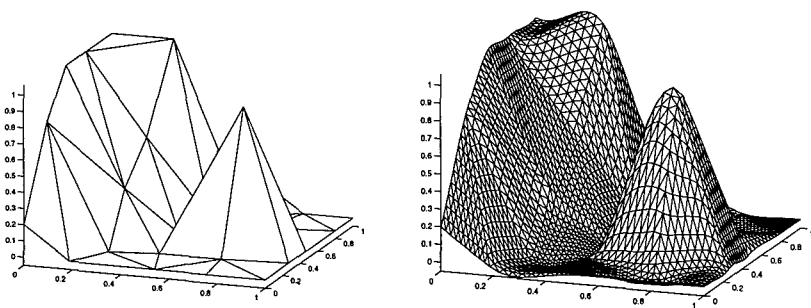


Fig. 6. *Left:* the Ritchie's Hill data. *Right:* the interpolating surface with uniform degrees $n_j = 3$.

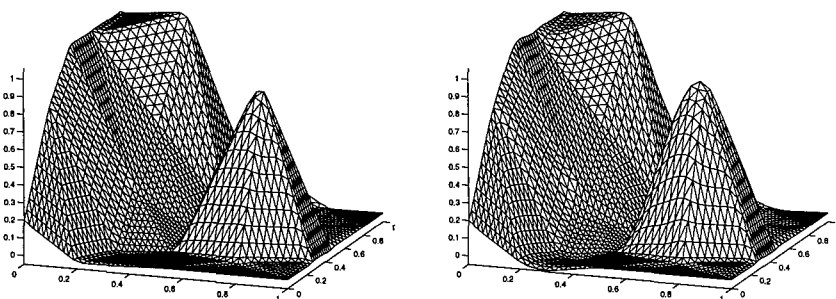


Fig. 7. *Left:* the interpolating surface with uniform degrees $n_j = 9$. *Right:* the interpolating surface with nonuniform degrees.

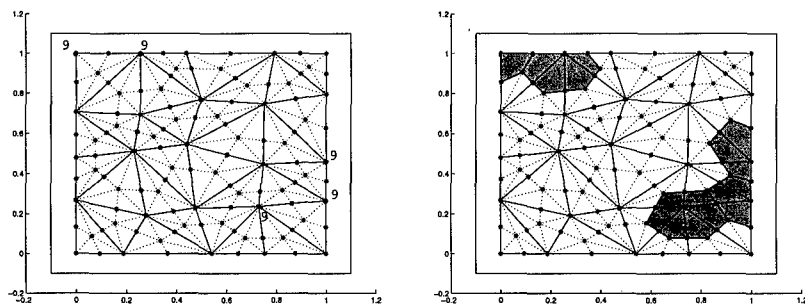


Fig. 8. *Left:* degrees $\neq 3$ depicted at the corresponding vertices. *Right:* the influence region of the increased degrees.

§6. Numerical Results

In this section we present a classical graphical example to show the performances of the macro-element and the local tension effect of the degrees.

The data (Fig. 6 left) are taken from [6] (see also Fig. 8 left for the considered triangulation). The interpolating surfaces obtained by using the proposed macro-element with uniform degrees $n_j = 3$ is depicted in Figure 6 right. Figure 7 left shows the interpolating surfaces obtained by using uniform degrees $n_j = 9$. The tension effect due to the increased values of the degrees is evident, but obviously uniformly distributed over all of the surface. On the other hand, the surface in Figure 7 right has been obtained considering all the degrees equal to 3, except those associated with the five vertices as depicted in Figure 8 left. The local tension effect of the degrees is clear. The influence region of the increased degrees can be also seen in Figure 8 right.

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